THE NUMERICAL SOLUTION OF ABEL'S INTEGRAL EQUATION*

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A method for the numerical solution of Abel's integral equation, based on the expansion of the unknown function in orthogonal polynomials, is presented.

Abel's integral equation

$$F(x)=\int_0^x \frac{f(y)\,dy}{\sqrt{(x-y)}}, \qquad 0\leqslant x\leqslant a, \tag{1}$$

in which $F(x)$ is given, and $f(y)$ is an unknown function, has a known exact solution (see [1, 2])

$$f(x)=\frac{1}{\pi}\frac{d}{dx}\int_0^x \frac{F(y)\,dy}{\sqrt{(x-y)}}.$$

However, in the case where the original function $F(x)$ is specified with an error, its direct use is limited, and in fact up to the present time there has been no sufficiently simple and efficient algorithm for the numerical solution of Abel's equation taking this error into account (see, for example, [3], where detailed references are given).

The optimal method of solving Abel's equation numerically (that is, the one giving the least loss of accuracy in comparison with the accuracy of the initial data) must take into account both the statistical properties of the function $F(x)$, connected with the errors of measurement, and also the analytic properties of Abel's transformation, expressed by Eq. (1).

In this paper we present a method of calculating the unknown function $f(x)$ which takes these factors into account. This method is based on the expansion of the unknown function $f(x)$ in eigenfunctions of the integral operator

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$$Af=\frac{1}{2\sqrt{x}}\int_0^x\frac{f(y)\,dy}{\sqrt{(x-y)}},$$

which are the power functions $f_n(x)=x^n$ for $n=0, 1, 2, \ldots$ (see [2]). The corresponding eigenvalues are calculated by the recurrence formula

$$\lambda_0=1, \qquad \lambda_n=\lambda_{n-1}/(1+1/2n), \qquad n=1, 2, \ldots .$$

Therefore, $f(x)=x^n$, implies that $F(x)=2\lambda_n x^n\sqrt{x}$. A difference of $F(x)$ from the latter expression can only be due to errors of measurement.

In all the physical applications of Abel's equation the unknown function $f(x)$ can be approximated with sufficient accuracy by a polynomial of finite degree

$$f(x)=\sum_{k=0}^{m} f_k x^k, \qquad 0\leqslant x\leqslant a. \tag{2}$$

In this case the experimental function $F(x)$ must be represented in the form

$$F(x)=\varphi(x)\sum_{k=0}^{m}\lambda_k f_k x^k, \tag{3}$$

where $\varphi(x)=2\sqrt{x}$.

The expansion (3) is found by the method of mean-square approximation by the functions

$$\Phi_k(x)=\varphi(x)P_k(x),$$

orthogonal in the given segment $0\leqslant x\leqslant a$ with weight

$$w(x)\sim 1/[\delta F(x)]^2,$$

where $\delta F(x)>0$ is the mean-square error of the measurement of the function F at the point x and $P_k(x)$ are orthogonal polynomials in the segment $[0, a]$ with the weight function $q(x)=w(x)\varphi^2(x)$.

Depending on how the function $F(x)$ is specified, different systems of polynomials are constructed. For a continuous definition of $F(x)$ and $w(x)=\text{const}$ these polynomials are the classical Jacobi polynomials $P_k^{(0,1)}((2x-a)/a)$. When $F(x)$ is specified at a discrete system of points $x_1, x_2, \ldots, x_n$ the corresponding orthonormal system of polynomials of degree not higher than $n-1$ is constructed (see [4]). The degree m of the approximating

polynomial is chosen by the method of regression analysis using the statistic

$$\chi^2 = \sum_{i=1}^{n} \left\{ \left[F(x_i) - \varphi(x_i) \sum_{k=0}^{m} c_k P_k(x_i) \right] [\delta F(x_i)]^{-1} \right\}^2,$$

which is distributed like χ^2_{n-m-1} (see [5]).

While determining the orthogonal polynomials, their coefficients are calculated and thereby the required expansion

$$F(x) = \varphi(x) \sum_{k=0}^{m} F_k x^k. \tag{4}$$

After this we calculate

$$f_k = F_k/\lambda_k$$

and by Eq. (2) we find $f(x)$. The corridor of errors of the recovered solution, arising due to random errors in the initial data, is calculated by the formula

$$\delta f(x) = 2 \left[\sum_{\alpha,\beta=0}^{m} (DF)_{\alpha\beta} x^\alpha x^\beta / (\lambda_\alpha \lambda_\beta) \right]^{1/2}. \tag{5}$$

In this expression DF is the covariance matrix (the error matrix) of the coefficients K_k determined in obtaining the expansion (4).

This corridor of errors, or the non-removable error, corresponds approximately to the 95% confidence zone for the solution $f(x)$ (see [5, 8]). In practical problems the principal source of the reconstruction error is the presence of random errors in the original data. Therefore, the estimate (5) of the error is inescapably of a probability nature.

It is obvious from (5) that the numerical stability of the proposed method is completely determined by the law of increase of the numbers $1/\lambda_m$ as the degree of approximation increases. From the asymptotic formula

$$1/\lambda_m = \sqrt{[2(2m+1)/\pi]} \qquad \text{if} \quad m \gg 1,$$

already giving an error of less than 5% for $m=3$ it can be established that for $5 \leqslant m \leqslant 10$ the value of $1/\lambda_m$, and consequently also the accuracy of the definition of the function $f(x)$ is only less by a factor of 3–3.5 than the accuracy with which the original function $F(x)$ is given.

The use of polynomials of finite degree is essentially a method of regularizing the solution of Abel's integral equation, enabling the growth of errors to be limited. This method is based on a well-known theorem of Picard (see [1]) on the expansion of the solution in mutually conjugate characteristic functions of the kernel.

We mention that the expansion of the function $F(x)$ in orthogonal polynomials in equation (1) used in [3, 6] instead of $f(y)$ as in the present paper, leads to a narrower class of reconstructed functions. For example, functions $f(y)\neq 0$ for $y=0$ (or $I(r)\neq 0$ for $r=R$ in Eq. (6), see below) are not permitted. We also notice that the problem could be solved by using stable methods of numerical differentiation based on the approximation of the function to be differentiated by polynomials of finite degree [7]. However, the method proposed by us is simpler, since it does not require a preparatory calculation of the integral.

As an example we consider two physical problems. The first refers to the physics of a plasma: in the measurement of the radial distribution of the radiative intensity of a cylindrical plasma column the equation

$$J(x)=2\int_0^{\sqrt{(R^2-x^2)}} I(\sqrt{(x^2+y^2)})\,dy=2\int_x^R \frac{I(r)\,r\,dr}{\sqrt{(r^2-x^2)}}, \qquad 0\leqslant x\leqslant R, \tag{6}$$

has to be solved, where $J(x)$ is the measured function, and $I(r)$ is the unknown function (see [6]). By the change of variables

$$\rho=R^2-r^2, \qquad i(\rho)=I(\sqrt{R^2-\rho})$$

Eq. (6) is reduced to Abel's equation for the functions J and i.

The following problem was solved to check the accuracy of the proposed method and compare it with previously known methods. For the two functions

$$I_1(r)=\begin{cases} 1-2r^2, & 0\leqslant r\leqslant 0.5,\\ 2(1-r)^2, & 0.5\leqslant r\leqslant 1\end{cases}$$

(this function has been used by a number of authors (see [3]) for checking the accuracy of a method of solving Abel's equation, and therefore it is suitable for comparison), and

$$I_2(r)=(1-r^2)^{1/2}+\exp\left[-\left(\frac{r-0.6}{0.3}\right)^2\right]+\exp\left[-\left(\frac{r+0.6}{0.3}\right)^2\right]+0.5$$

the integral $J(x)$ was evaluated by Eq. (6) with an accuracy up to 10^{-5} for 51 values of x from 0 to 1 with a step of 0.02. To the values obtained were added random normally distributed "errors of measurement" with zero mean value and variance σ. (The tables of random numbers used are given in [8].) From these data the approximate solutions $\hat{f}_1(r)$ and $\hat{f}_2(r)$. were obtained. The calculations were carried out on the BESM-6 computer; the time for computing the first version was ~4 sec. The results of the reconstruction for $\sigma = 2 \cdot 10^{-4}$ are shown in Table 1.

TABLE 1

r	$I_1(r)$, the exact solution	$\hat{f}_1(r)$, the given method	$\hat{f}_1(r) - I_1(r)$, the actual error of reconstruction	The estimate of the error of reconstruction from (5)	The error of reconstruction by the method described in [3] for $\sigma = 10^{-4}$
0	1.000	1.0040	+0.0040	0.0020	+0.0012
0.1	0.980	0.9811	+0.0011	0.0012	0
0.2	0.920	0.9185	−0.0015	0.0007	−0.0006
0.3	0.820	0.8207	+0.0007	0.0006	+0.0010
0.4	0.680	0.6799	−0.0001	0.0006	−0.0006
0.5	0.500	0.4999	−0.0001	0.0005	+0.0003
0.6	0.320	0.3199	−0.0001	0.0005	−0.0001
0.7	0.180	0.1800	0	0.0006	−0.0004
0.8	0.080	0.0799	−0.0001	0.0007	+0.0007
0.9	0.020	0.0207	+0.0007	0.0009	−0.0002
0.95	0.005	0.0049	−0.0001	0.0010	+0.0023
1.0	0	0.0058	+0.0058	0.0062	0

It is obvious from the data presented that the proposed method has approximately the same accuracy as the best of the previously known methods [3]. However, as mentioned above, in the proposed method it is not obligatory that $I(r) = 0$ for $r = R$, as is required in [3], and therefore it has a wider field of application. The estimate of the reconstruction error by Eq. (5) is fairly close to the actual error.

Figure 1 shows the results of the approximated function $\hat{f}_2(r)$ for the errors of measurement $\sigma = 0.1$. It is obvious from Figure 1 that the method is also applicable to these (comparatively large) errors in the original data; the recovered function reproduces the features of the solution, and the exact solution lies almost completely within the corridor of errors of recovery calculated by Eq. (5).

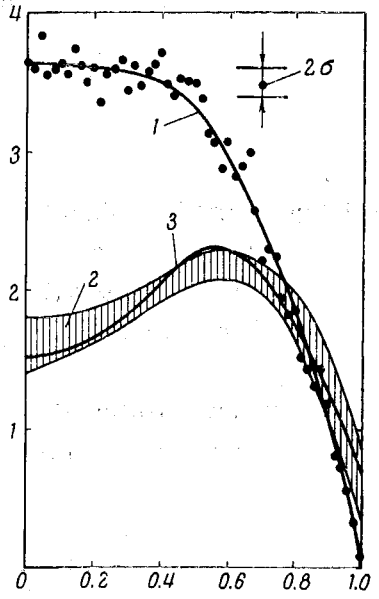

FIG. 1

● are the original data; *1* is the approximation of the original data by the expression (4) for $m = 2$; *2* is the corridor of errors of the reconstructed solution; *3* is the exact solution.

The second problem relates to physical electronics. In the processing of data from measurements of the emission in a high-frequency field it is required to solve the equation

$$J(E)=\frac{1}{\pi}\int_0^{\pi/2} I(E\sin\varphi)\,d\varphi, \tag{7}$$

in which $J(E)$ is the measured, and $I(E)$ the unknown function [9]. By the change of variables

$$E=\sqrt{x}, \qquad E\sin\varphi=\sqrt{y}, \qquad I(\sqrt{y})/\sqrt{y}=f(y), \qquad J(\sqrt{x})=F(x)$$

Eq. (7) can be reduced to Abel's equation. However, it is more convenient to solve Eq. (7) directly by expanding the solution $I(E)$ in eigenfunctions of the integral operator

$$LI=\frac{1}{\pi}\int_0^{\pi/2} I(E\sin\varphi)\,d\varphi,$$

which are also power functions E^n for $n=0, 1, 2, \ldots$. The eigenvalues μ_n of the operator L are calculated from the relations

$$\mu_0=1/2,\ \mu_1=1/\pi,$$
$$\mu_n=\mu_{n-2}(1-1/n),$$
$$n=2,\ 3,\ \ldots.$$

The calculations carried out show that the accuracy and stability of this method of numerical solution of the integral equation (7) correspond mainly to the characteristics for the method of numerical solution of Abel's integral equation explained above. The method described was used in practice for measuring the cathode loss in a microtron [9].

In conclusion we mention that the proposed method for the numerical solution of Abel's integral equation has a natural generalization for the equation

$$F(x)=\int_0^x \frac{f(y)\,dy}{(x-y)^\alpha}, \qquad 0<\alpha<1,$$

with an arbitrary index α.

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