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APPLICATIONS OF INTEGRAL EQUATIONS OF THE FIRST KIND IN EXPERIMENT PHYSICS

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Applications to plasma diagnostics, physical electronics, nuclear physics, star cluster problem and super-resolution processing improvements in optical imaginary are presented. The computer program implementation of inverse problems are discussed as well.

1. Introduction

This report consists of two parts. The first part is based on the author's work and describes applications of Abel's integral equation to some problems in experimental plasma physics, electronics and astronomy. The second one is based on some papers published in J. Opt. Soc. Am., Nucl. Instr. and Meth. and Proc. IEEE and is a review of a very prominent method of one-dimensional function and two-dimensional image restoration that has simultaneously many different names: maximum likelihood, maximum entropy, minimum-directioned discrepancy and Bayesian deconvolution. But it is the same method, and some applications and examples in experimental nuclear physics and astronomy will be discussed in the second part.

2. Applications of the Abel's integral equation

To determine the radial distribution of the radiation intensity of a cylindrical discharge in plasma physics one should solve the first-kind integral equation

$$J(\xi) = 2\int_0^{\sqrt{R^2-\xi^2}} I(\sqrt{\xi^2+\eta^2})\,\mathrm{d}\eta$$

$$= 2\int_\xi^R \frac{I(r)\,r\,\mathrm{d}r}{\sqrt{r^2-\xi^2}}, \quad 0 \leqslant \xi \leqslant R. \tag{1}$$

where $J(\xi)$ is the measured function and $I(r)$ is the one to be determined, see fig. 1. By the change of variables

$$x = R^2 - \xi^2, \quad y = R^2 - r^2,$$

$$F(x) = J(\sqrt{R^2-x}), f(y) = I(\sqrt{R^2-y})$$

eq. (1) reduces to the standard Abel integral equation for functions F and f

$$F(x) = \int_0^x \frac{f(y)\,\mathrm{d}y}{\sqrt{x-y}}. \tag{2}$$

Although this equation has a well-known exact solution

$$f(x) = \frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{F(y)\,\mathrm{d}y}{\sqrt{x-y}}, \tag{3}$$

some difficulties arise in using the exact solution (3) in practice. The point is that the function $F(x)$ (or $J(\xi)$) is known from measurements with only a finite

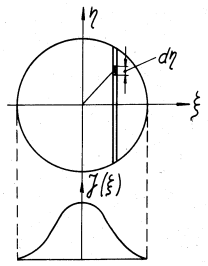

Fig. 1. Geometry of eq. (1).

(usually not good) precision and therefore large uncertainties of the unknown function $f(x)$ appear which are bigger than those in $F(x)$. This is a general feature of so-called "incorrect" or "ill-posed" problems. Integral equations of the first kind with smooth kernels lead to such problems.

But we do not have a smooth kernel $K(x, y) = 1/\sqrt{x-y}$. This fact and use of the known statistics of measurement noise give us an adequate method for solving the problems (1) or (2). We call it the "orthogonal expansion method" (OEM) and it may be applied for various problems of restoration. The idea is the following: we represent the unknown function $f(x)$ (but not the experimental function $F(x)$!) as a sum of some suitable basis functions

$$f(x) = \sum_{\alpha} c_\alpha \varphi_\alpha(x) .$$

As a result we have the expansion

$$F(x) = \sum_{\alpha} c_\alpha \psi_\alpha(x) ,$$

where $\psi_\alpha(x) = A\varphi_\alpha(x)$ and A is the integral operator

$$Az(x) = \int_0^x \frac{z(y)\,dy}{\sqrt{x-y}} .$$

The problem now reduces to determining the coefficients c_α from measurements. It is convenient for this purpose to introduce a new orthogonal basis $\{e_\alpha\}$: $(e_\alpha, e_\beta) = \delta_{\alpha\beta}$ (round brackets mean the scalar product), which is linearly connected with the basis $\{\psi_\alpha\}$

$$e_\alpha(x) = \sum_{\beta} u_{\alpha\beta} \psi_\beta(x) ,$$

and the matrix $\|u_{\alpha\beta}\|$ is lower triangular: $u_{\alpha\beta} = 0$ for $\beta > \alpha$. If we find the expansion

$$F(x) = \sum_{\alpha} s_\alpha e_\alpha(x)$$

then we shall know the coefficients

$$c_\alpha = \sum_{\beta} u_{\beta\alpha} s_\beta .$$

The coefficients s_α we find by the recurrence formula

$$s_\alpha = \left(F - \sum_{\beta=0}^{\alpha-1} s_\beta e_\beta, e_\alpha\right)$$

rather than the standard one $s_\alpha = (F, e_\alpha)$.

This is one of the small computing tricks that provide the high performance of the method.

Now we will show how we use information about the statistics of measurement noise to determine the number of terms in each of the above sums. The quantity

$$\chi^2 = \sum_{i=1}^{N} \left[F(x_i) - \sum_{\alpha=0}^{M} s_\alpha e_\alpha(x_i)\right]^2 / \sigma^2[F(x_i)]$$

has a chi-squared distribution with $N - M - 1$ degrees of freedom. In this formula x_i ($i = 1, 2, \ldots, N$) means the experimental points at which the measurements were made, and $\sigma^2[F(x_i)]$ is the variance of measurement noise (measurement errors) at the points x_i. According to the Pearson criterion the number of terms $M + 1$ should be increased if

$$\chi^2 > \chi^2_{N-M-1}(P) ,$$

where P is the significance level of the criterion. We usually adopt $P = 5\%$. The Fischer criterion may be applied instead of the Pearson criterion if $\sigma^2[F(x_i)]$ is unknown and in this case only the estimate $\hat{\sigma}^2[F(x_i)]$ is used. Using only a finite number of terms in each of the above sums is an extremely important feature of the orthogonal-expansion method. This is the end of our brief description of the OEM.

The basis functions $\varphi_\alpha(x) = x^\alpha$ ($\alpha = 0, 1, 2, \ldots$) were chosen by the author in ref. [1]. This basis corresponds to a polynomial approximation of the un-

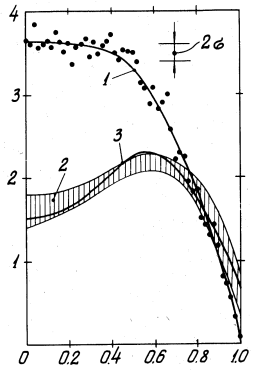

Fig. 2. Test example for eq. (1); the points are input data; 1 – approximations of input data, 2 – uncertainties of solution $I(x)$, 3 – the solution $I(x)$.

known function $f(x)$. The test example is shown in fig. 2. As is seen in this figure the method is usable even for large measurement noise of the input data. In ref. [2] it was shown how eq. (1) should be applied in plasma diagnostics with strong refraction.

The next example deals with physical electronics [3]. For the study of electron emission from cathodes in the alternating high-frequency electric field of the electron accelerator-microtron one requires to solve the integral equation

$$J(E) = \frac{1}{\pi} \int_0^{\pi/2} I(E \sin \varphi)\, \mathrm{d}\varphi\,, \qquad (4)$$

where $J(E)$ is the measured function, and $I(E)$ is the one to be determined. This equation is known as the Schlömilch integral equation and by a change of variables it may be reduced to the Abel integral equation. But we may apply the OEM for this equation once again. The basis functions may be $\varphi_\alpha = E^\alpha$ ($\alpha = 0, 1, 2, \ldots$) too. The input and output data of eq. (4) for a LaB_6 cathode are shown in figs. 3 and 4.

In astronomy there is an interesting problem of determining the inner structure of globular star clusters. Mathematically this reduces to an integral equation that is absolutely identical to eq. (1), but the meanings of functions J and I differ from those in the plasma-physics problem. In astronomy $I(r)$ is the three-dimensional density of star numbers per

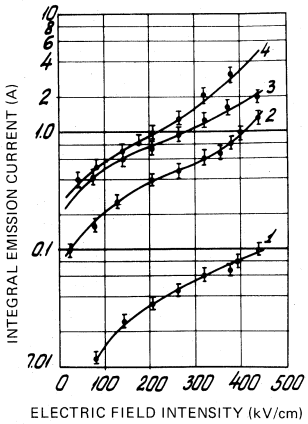

Fig. 3. Input data $J(E)$ for eq. (4), LaB_6 cathode, 2.5 × 2.5 mm^2; 1 – T = 1600°C, 2 – 1650°C, 3 – 1680°C, 4 – 1720°C.

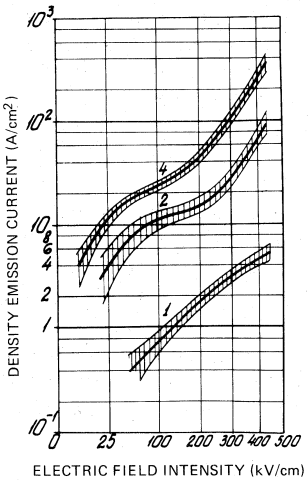

Fig. 4. Output data $I(E)$ for eq. (4) 1 to 4 – the same as fig. 3.

unit volume, and $J(R)$ is the surface density of star numbers per unit of sky area. If the star cluster does not resolve into separate stars and we can observe from earth only the integral star luminosity then the solution of the astronomical problem is totally identical to that in plasma physics.

The case, when we can observe from the earth two sky coordinates of each star in the cluster, is more interesting but more difficult for solution. For example, fig. 5 shows the cluster of flare stars in the Pleiades in projection on the sky surface at radius 126 ps [4]. Before beginning the known procedure of solution by OEM we should determine the surface density $J(R)$ from coordinate data. It is wonderful that OEM is applicable to this problem once again!

On the assumption that the cluster has spherical symmetry it is required to reconstruct the one-dimensional probability density $p(x)$ from observational data $x_1, x_2, \ldots, x_N$. This is the classical problem of mathematical statistics. By OEM we look for an approximation to the probability density

$$p(x) = \sum_\alpha c_\alpha \varphi_\alpha(x)\,,$$

where $\{\varphi_\alpha(x)\}$ is a suitable orthogonal basis

$$(\varphi_\alpha, \varphi_\beta) = \int \varphi_\alpha(x)\, \varphi_\beta(x)\, \mathrm{d}x = \delta_{\alpha\beta}\,.$$

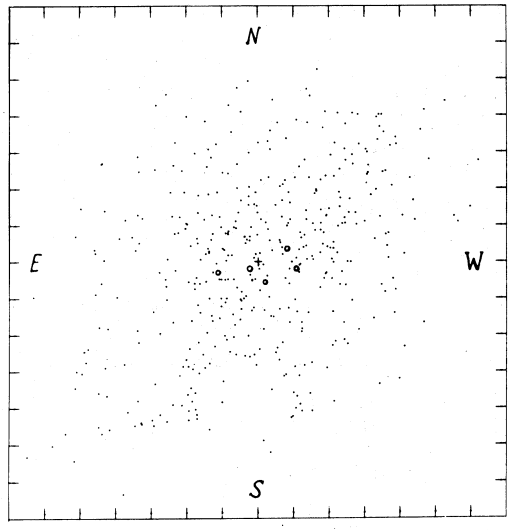

Fig. 5. Cluster of flare stars in the Pleiades. Small points are flare stars, small circles are five of the brightest Pleiades stars. Scale is 1 ps per division.

If the probability density $p(x)$ were known we should have the result

$$c_\alpha = (p, \varphi_\alpha) = \int p(x)\, \varphi_\alpha(x)\, dx .$$

But $p(x)$ is unknown and it seems that we have a vicious circle. However, we have a good idea. Since the $p(x)$ is a probability density the last formula is the expression for the mean of $\varphi_\alpha(x)$

$$\int p(x)\, \varphi_\alpha(x)\, dx = \overline{\varphi_\alpha}$$

and we use for the mean of $\varphi_\alpha(x)$ the arithmetical mean estimation

$$\hat{\overline{\varphi_\alpha}} = \frac{1}{N}\sum_{i=1}^{N} \varphi_\alpha(x_i) = \hat{c}_\alpha ,$$

where $x_1, x_2, \ldots, x_N$ are independent observational data, each of them having the same probability density $p(x)$. This is the crucial step in this reasoning!

The estimation of $\hat{c}_\alpha$ is unbiased:

$$\overline{\hat{c}_\alpha} = \frac{1}{N}\sum_{i=1}^{N} \overline{\varphi_\alpha(x_i)} = \frac{1}{N}\sum_{i=1}^{N} c_\alpha = c_\alpha ,$$

and consistent:

$$D(\hat{c}_\alpha \hat{c}_\beta)$$

$$= \frac{1}{N}\int (\varphi_\alpha(x) - c_\alpha)(\varphi_\beta(x) - c_\beta) p(x)\, dx \to 0$$

at $N \to \infty$.

As all the $\hat{c}_\alpha$ are sums of a large number of terms $N >> 1$ with the same probability distributions, then, according to the central limit theorem of probability theory, all the estimates $\hat{c}_\alpha$ jointly have normal distributions with means c_α and covariance matrix $D(\hat{c}_\alpha \hat{c}_\beta)$.

We now have all the information not only to determine $\hat{c}_\alpha$, but to choose the number of terms $M + 1$ in the approximation of $p(x)$

$$p(x) \approx \sum_{\alpha=0}^{M} \hat{c}_\alpha \varphi_\alpha(x) .$$

In this formula we use only those coefficients $\hat{c}_\alpha$ which have

$$\hat{c}_\alpha^2 / \hat{D}(\hat{c}_\alpha \hat{c}_\beta) > 1 .$$

This condition provides the minimum of the mean-squared approximation error: $\min \overline{R}_M$, where

$$R_M = \int \left[p(x) - \sum_{\alpha=0}^{M} \hat{c}_\alpha \varphi_\alpha(x) \right]^2 dx .$$

Instead of this criterion of choice M we may use the standard Fisher criterion, or as $N >> 1$ the Pearson criterion

$$\hat{c}_\alpha^2 / \hat{D}(\hat{c}_\alpha \hat{c}_\alpha) > \chi^2_{N-1}(p) , \qquad (p = 0.05) .$$

Usually the results are independent of the choice of different criteria.

Chenzov was the first to propose the OEM for the probability-density estimation in 1962 [5]. This method has a precision of order $N^{-1/2}$ and in 1978 Boyd and Steele proved that there is no other density estimator which has a better precision [6].

Now we come back to the star problem. At this point we have the estimate of surface star density $J(R)$ and we can just use the above-described algorithm for the solution of Abel's equation. In fig. 6 the volume density of flare stars in the Pleiades cluster computed by the author is shown. One can

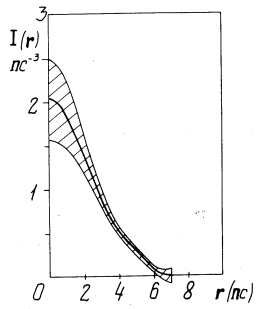

Fig. 6. The volume density of flare stars in the Pleiades.

see in this figure that the density has a maximum in the centre of the cluster.

3. Maximum likelihood, minimum-directioned discrepancy, Bayesian deconvolution and maximum-entropy methods in restoration problems

Strictly speaking, there is always noise of various origins and in restoration problems we have the equations in the form

$$F(x) = \int K(x, y)\, g(y)\, \mathrm{d}y + N(x)\,, \tag{5}$$

where the noise $N(x)$ is known only statistically, rather than in the form (1), (2) or (4).

In many applications, e.g. in experimental nuclear physics or in photographic-image restoration, both the detected function $F(x)$ and the unknown one $g(y)$ are non-negative:

$$F(x) \geqslant 0\,, \qquad g(y) \geqslant 0\,.$$

In such cases we may consider that $F(x)$ and $g(x)$ are probability functions with unit norm

$$\int F(x)\, \mathrm{d}x = \int g(x)\, \mathrm{d}x = 1$$

and we use the powerful maximum-likelihood method (MLM) for restoration of unknown object $g(x)$ from observations of $F(x)$.

For simplicity we shall use the discrete form of (5)

$$f_i = \sum_k p_{ik} g_k + n_i\,, \quad i = 1, 2, \ldots, m_1\,, \quad k = 1, 2, \ldots, m_2. \tag{5'}$$

In to-day's experimental physics the measurements are usually carried out with a multichannel analyser and it will be convenient to consider $N_i = Nf_i$ as the independent observational data in the ith channel and $N = \Sigma_i N_i$ as the total number of collected data. In this language the probability P to receive from observations the data set $\{N_i\}$, which totally coincides with the actual set, is equal to

$$P = (N!/\prod_i N_i!) \prod_i p_i^{N_i}\,,$$

where $p_i = \Sigma_k p_{ik} g_k$. According to the MLM philosophy the maximization of P or $\ln P$ gives the equation for determination of the unknown object $\{g_k\}$:

$$\max_{\{g_k\}} (\ln P) = \max_{\{g_k\}} (\text{const} + N \sum_i f_i \ln \sum_k p_{ik} g_k) \,. \tag{6}$$

This maximum always exists, as there is the known information-theory inequality

$$\sum_i f_i \ln \sum_k p_{ik} g_k - \sum_i f_i \ln f_i = \sum_i f_i \ln(p_i/f_i) \leqslant 0\,.$$

The second term $\Sigma_i f_i \ln f_i$ is constant and therefore does not depend on $\{g_k\}$.

Another way to get eq. (6) is to minimize the directioned discrepancy (MDD) between two probability functions $\{f_i\}$ and $\{\Sigma_k p_{ik} g_k\}$, which is equal to $\Sigma_i f_i \ln(f_i/p_i)$ (see ref. [7]). The sense of such minimizing is to adjust as close as possible the measured function $\{f_i\}$ and the unknown one $\{\Sigma_k p_{ik} g_k\}$. Since $\Sigma_i f_i \ln(f_i/p_i) = -\Sigma_i f_i \ln(p_i/f_i)$ the MLM and MDD method are equivalent.

How to find the maximum in (6)? An approach based on Bayes's theorem has been proposed by Tarasko [8] in 1969 for nuclear-physics applications and by Richardson [9] in 1972 for image restoration. As we consider the $\{f_i\}$ and $\{g_k\}$ are probability functions $\Sigma_i f_i = \Sigma_k g_k = 1, f_i \geqslant 0, g_k \geqslant 0$, and $\{p_{ik}\}$ is a stochastic matrix $\Sigma_i p_{ik} = 1, p_{ik} \geqslant 0$ for all i and k, then the approximate solution of eq. (5'), satisfying the condition $\Sigma_k g_k = 1, g_k \geqslant 0$ may be constructed by the iterative procedure

$$g_k^{(s+1)} = g_k^{(s)} \sum_i p_{ik} f_i / \sum_j p_{ij} g_j^{(s)}\,, \tag{7}$$

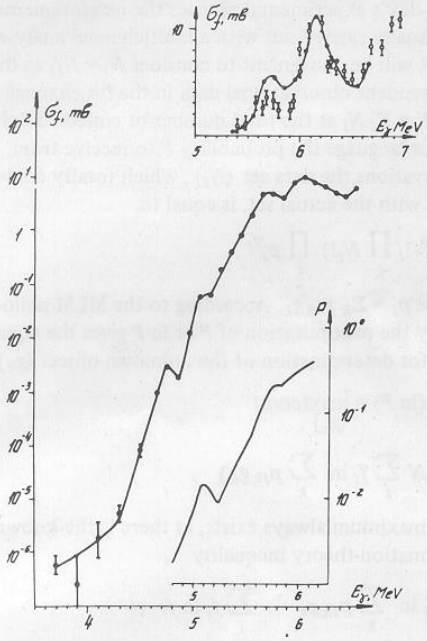

Fig. 7. The photofission cross-section of ^{238}U [10].

where s is the iteration index. This procedure automatically provides non-negativity: $g_k \geqslant 0$. Tarasko has proved that this iterative procedure gives the solution of the MDD function $\Sigma_i f_i \ln(f_i/p_i)$ and therefore the three methods: maximum likelihood, minimum-directioned discrepancy and Bayesian deconvolution are equivalent.

This method has been successfully used in the "bremsstrahlung" experiment by Zhuchko and Tsipenyuk [10]. In fig. 7 the results of their sub-threshold photofission experiment of ^{238}U are shown in comparison with the independent data received by direct experiments without unfolding. There is a reasonable agreement between them. The paper of Zhuchko [11] deals with important modifications of the iterative procedure (7) to reduce computation time and to improve its accuracy. A series of papers on this method was published recently by Kennet et al. [12].

The maximum-entropy method (MEM) in restoration problems is very similar to MLM and MDD. It starts from the papers of Jaynes [13] and Frieden [14]. There is a new review of this method by Keating et al. [15]. In this method the solution of eq. (5′) involves maximizing the sum of the unknown object

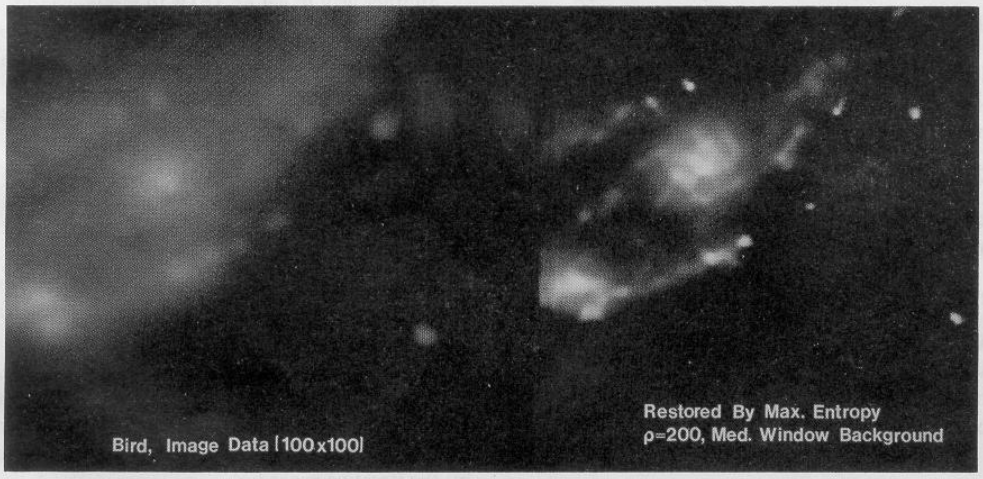

Fig. 8. Example of the maximum-entropy method. Left – the original photopicture, right – MEM restoration (result of Frieden [15]).

entropy $H_g = -\Sigma_k g_k \ln g_k$ and the noise entropy $H_n = -\Sigma_i n_i \ln n_i$ under constraints (5′), so that the results are consistent with the measured data. The result of the maximization problem is

$$g_k = \exp\left(-1 - \sum_i \lambda_i p_{ik}\right), \qquad n_i = \exp(-1 - \lambda_i), \tag{8}$$

where λ_i are the Lagrange multipliers determined from the constraint equations (5′). The solution of system (8) and (5′) may be done only by a computer, e.g. by using the Newton–Raphson method.

A striking example from astronomical optics is shown in fig. 8.

4. Conclusion

The orthogonal-expansion method (OEM) for the solving of restoration problems applies only to linear equations. The Bayesian deconvolution is very effective and very simple to realize on a computer, but is again applicable only to linear equations. The maximum-likelihood and maximum-entropy methods are suitable not only for linear but for non-linear equations as well although they are more expensive in computer realization and time.

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